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RIGIDITY AND DEFORMATION SPACES OF STRICTLY CONVEX REAL PROJECTIVE STRUCTURES ON COMPACT MANIFOLDS

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To my child

Abstract

In this paper we show that if two strictly convex, compact real projective manifolds have the same marked length spectrum with respect to the Hilbert metric, then they are projectively equivalent. This is a rigidity for Finsler metric with a special geometric structure. Furthermore we prove an analogue of a Hitchin's conjecture for hyperbolic 3-manifolds, namely the deformation space of convex real projective structures on a compact hyperbolic 3-manifold M is a component in the moduli space of $\mathrm{PGL}(4, \mathbb{R})$-representations of $\pi_1(M)$.

1. Introduction

In the Riemannian case, it is conjectured that two compact negatively (or nonpositively) curved manifolds are isometric if they have the same marked closed geodesic lengths. The conjecture is true for surfaces [16, 41]. If one manifold is negatively curved locally symmetric, then the conjecture is true by [7, 29]. If one manifold is of rank at least two, it is proved by [17]. When both manifolds are locally symmetric but are of infinite volume, see [33, 19]. But in the non-Riemannian case the situation is much more complicated. In this paper we restrict the problem to real projective manifolds equipped with Hilbert metrics. Since the manifolds have a real projective geometric structure other than the Finsler metric we can use the extra information to attack the problem.

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The main results we want to prove are:

Theorem I. *Let M and N be compact, strictly convex real projective manifolds with Hilbert metrics. If they have the same marked length spectrum then they are projectively equivalent.*

The other theorem is an analogue of a Hitchin's conjecture [30] in dimension 3. The Teichmüller component is defined to be the component of a representation variety $\mathfrak{R}(\pi_1(M), \mathrm{PGL}(n, \mathbb{R}))$ which contains the hyperbolic holonomy representations where the = representation variety is $\mathrm{Hom}(\pi_1(M), \mathrm{PGL}(n, \mathbb{R}))/\mathrm{PGL}(n, \mathbb{R})$. Denote by $\mathfrak{B}(M)$ the set of strictly convex real projective structures on the manifold M and $\mathfrak{B}^0(M)$ a component of $\mathfrak{B}(M)$ containing hyperbolic structures.

Theorem II. *The holonomy map*

$$h : \mathfrak{B}^0(M) \to \mathfrak{R}(\pi_1(M), \mathrm{PGL}(4, \mathbb{R}))$$

is a homeomorphism onto the Teichmüller component if M is a hyperbolic 3-manifold. Here h is a map associating each convex real projective structure to its holonomy representation.

Note that due to [31] (Section 5), $\mathfrak{B}^0(M)$ has positive dimension for certain hyperbolic manifolds M. Using Theorem I we give a corollary.

Corollary 0. *Let $M = C_1/\Gamma_1$ and $N = C_2/\Gamma_2$ be compact strictly convex real projective n-manifolds. Then there exists a cross-ratio preserving equivariant homeomorphism between ∂C_1 and ∂C_2 if and only if M and N are projectively equivalent.*

This type of theorem is known between a negatively curved locally symmetric manifold and a quotient of a $\mathrm{CAT}(-1)$ space [9, 34].

Plan of the paper. Theorem I is proved in Section 8 where Proposition 2 and Theorem 2 give an algebraic proof of it. Corollary 0 is proved in Section 9 where cross-ratio and other techniques are developed. Theorem II is proved in Section 10. Theorem 1 in Section 4 describes the general limit of nonparabolic discrete faithful representations of a centerless, nonsolvable group in $\mathrm{PSL}(n, \mathbb{R})$. To prove this we use some techniques in symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$. Proposition 6 in Section 10 shows that if such representations come from the holonomy of convex real projective structures on a closed hyperbolic 3-manifold, the limit representation is nonparabolic also. So the limit real projective structure does not degenerate. Using these, Theorem 4 gives a final proof of Theorem II.

2. Preliminaries

Let x_1, x_2, x_3, x_4 be four distinct points in $\mathbb{R}^n$.

Definition 1. The **cross-ratio** of four points is defined by

$$[x_1, x_2, x_3, x_4] = \frac{|x_1 - x_3||x_2 - x_4|}{|x_1 - x_4||x_2 - x_3|}$$

where $|\ |$ is the Euclidean metric.

In 1894 D. Hilbert discovered a generalization of the hyperbolic geometry, for which geodesics are still Euclidean segments ([13] IV. 28, [11, 12]). Let Ω be a bounded convex domain in $\mathbb{R}^n$.

Definition 2. Let x, y be two points in Ω. The Hilbert distance between x and y is:

$$d(x, y) = \log\,[x^*, y^*, y, x]$$

where x^*, y^* are on the boundary of Ω which lie on the line joining x and y such that x is between x^* and y.

If we add $d(x, x) = 0$, then this is a complete metric which induces the same topology in $\mathbb{R}^n$. For the proofs, see [10, 2]. Suppose C is strictly convex. This Hilbert metric is known to be Finslerian [36, 37] with the following properties:

(1) There is a unique geodesic between two points in Ω. See [29].

(2) There is a unique geodesic between two points in the boundary of Ω.

(3) The geodesics are straight lines in Euclidean sense.

If C is strictly convex, the set of points at positive distance from a convex set is strictly convex, so the spheres are convex [10]. The associated Finsler metric is not degenerate if the boundary has a nondegenerate Hessian and in this case the flag curvature is a negative constant [22, 21]. Benzécri [6] proved that this does not admit a compact quotient unless it is an ellipsoid.

3. Real projective structure

A real projective structure on a differentiable manifold M is a maximal atlas $\{U_i, \phi_i\}$ into $\mathbb{RP}^n$ such that the transition functions $\phi_j \circ {\phi_i}^{-1}$ are restrictions of projective automorphisms of $\mathbb{RP}^n$.

Definition 3. A **convex real projective** manifold M is Ω/Γ where Ω is a convex domain in $\mathbb{RP}^n$ containing no projective line and Γ is a discrete group of $\mathrm{Aut}(\mathbb{RP}^n)$.

Note that S^n inherits the projective structure from the double covering map of S^n onto $\mathbb{RP}^n$. Then we can situate Ω in the upper-hemisphere and project it onto the hyperplane $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n | x_n = 1\}$.

In general, a projective structure gives rise to a map, called a developing map and denoted dev, from the universal cover $\widetilde{M}$ of M to $\mathbb{RP}^n$ and a holonomy homomorphism h from $\pi_1(M)$ to $\mathrm{PGL}(n+1, \mathbb{R})$ so that the following diagram commutes.

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"] & \mathbb{RP}^n \arrow[d, "h(\gamma)"] \\ \widetilde{M} \arrow[r, "\mathrm{dev}"] & \mathbb{RP}^n \end{tikzcd}$$

The developing map is unique up to post-composition by an element g in $\mathrm{PGL}(n+1, \mathbb{R})$ and correspondingly the holonomy homomorphism is unique up to conjugate by g. If the developing map is a homeomorphism onto its image, then $M = \mathrm{dev}(\widetilde{M})/h(\pi_1(M))$.

Two projective manifolds M and N are considered to be equal if there exists a projective isomorphism between them, i.e., a diffeomorphism which is locally in $\mathrm{PGL}(n+1, \mathbb{R})$.

Formally a deformation space of $\mathbb{RP}^n$ structures on a fixed manifold M is defined as follows, see [24]. Let S be a fixed smooth manifold. Let $x \in S$ and $\widetilde{S}$ be a fixed universal covering of S. Then the set of triples (M, f, ψ) where M is an $\mathbb{RP}^n$ manifold, f is a diffeomorphism from S to M and ψ is a projective germ at $f(x)$, is equivalent to the set of development pairs (dev, h).

We mod out the set of triples up to equivalence relation that

$$(M_1, f_1, \psi_1) \sim (M_2, f_2, \psi_2)$$

iff there exist $\mathbb{RP}^n$ isomorphism $\phi : M_1 \to M_2$ such that $\phi \circ f_1$ is isotopic to f_2 by an isotopy fixing x and $\phi^*(\psi_2) = \psi_1$. Denote $\mathfrak{D}(S)$ the set of equivalence classes of such triples. Using the C^1 topology on developing maps, we give the set $\mathfrak{D}(S)$ a topology which is Hausdorff. Then the $\mathrm{PGL}(n, \mathbb{R})$-equivariant continuous map

$$\mathrm{hol} : \mathfrak{D}(S) \to \mathrm{Hom}(\pi_1(S), \mathrm{PGL}(n, \mathbb{R}))$$

is a local homeomorphism. We will call this the **holonomy theorem**. See [43] (Proposition 5.1), [40] and [24]. We denote $\mathbb{RP}^n(S) = \mathfrak{D}(S)/\mathrm{PGL}(n+1,\mathbb{R})$ the set of real projective structures on S. Denote $\mathfrak{B}(S) \subset \mathbb{RP}^n(S)$ the set of convex real projective structures and $\mathfrak{B}^0(M)$ a component of $\mathfrak{B}(M)$ containing hyperbolic structures. When $n = 3$, Goldman [25] showed that the set of convex $\mathbb{RP}^2$ structures on a closed surface M with $\chi(M) < 0$ is homeomorphic to an open ball of dimension $-8\chi(M)$ in $\mathrm{Hom}(\pi_1(M), \mathrm{PGL}(3,\mathbb{R}))/\mathrm{PGL}(3,\mathbb{R})$ by the map which associates a projective structure to its holonomy representation.

Suppose $M = C_1/\Gamma_1$ and $N = C_2/\Gamma_2$ are compact convex real projective manifolds such that Γ_1 and Γ_2 are conjugate by an element g in $\mathrm{PGL}(n,\mathbb{R})$. Then $g(C_1) = C_2$ and g descends to a projective isomorphism between M and N. We will use this fact to prove the main theorem.

If we denote $\mathrm{SL}_-(n,\mathbb{R})$ to be the set of matrices of determinant -1, then $\mathrm{PGL}(n,\mathbb{R}) = \mathrm{PSL}_-(n,\mathbb{R}) \cup \mathrm{PSL}(n,\mathbb{R})$ where $\mathrm{PSL}(n,\mathbb{R})$ denotes $\mathrm{SL}(n,\mathbb{R})/\pm I$ when n is even. Note that $\mathrm{PSL}(n,\mathbb{R})$ is the identity component of the isometry group of the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. In this paper, we will very often take a subgroup of index two so that the holonomy representation lies in $\mathrm{PSL}(n,\mathbb{R})$.

Let $\Gamma = \pi_1(M)$ be fixed and denote R the space of representations $\mathrm{Hom}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$. It is well-known that R is an algebraic variety in $\mathrm{PSL}(n,\mathbb{R})^k$ where k is the number of generators of Γ. In this note we are interested in R_{ir} the subset of irreducible representations. We just record the following lemma about the conjugate action of $\mathrm{PSL}(n,\mathbb{R})$ on R_{ir}.

Lemma 1. *The conjugate action of* $\mathrm{PSL}(n,\mathbb{R})$ *on* R_{ir} *is proper and free.*

Proof. Let $\mathrm{ev} : R_{ir} \to \mathrm{PSL}(n,\mathbb{R})^k$ be an evaluation map on the set of generators. Denote the embedding of R_{ir} in $\mathrm{PSL}(n,\mathbb{R})$ by U.

First we prove the action is free. If $g\rho g^{-1} = \rho$, since ρ is irreducible, g should be in the center of $\mathrm{PSL}(n,\mathbb{R})$ which is the identity.

Now we show that the action is proper. Take the Cartan decomposition $\mathrm{SL}(n,\mathbb{R}) = KA^+K$ where $K = \mathrm{SO}(n)$ and A^+ is the set of positive real diagonal matrices $(\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_1 \cdots \lambda_n = 1$. Then we have to show that for any compact set $C \subset U$ the set $G(C) = \{g \in \mathrm{PSL}(n,\mathbb{R}) | gCg^{-1} \cap C \neq \emptyset\}$ is compact. It is obvious that we can take C to be K invariant by enlarging it. Then since $k_1 a k_2 C k_2^{-1} a^{-1} k_1^{-1} = k_1 a C a^{-1} k_1^{-1} \cap C \neq \emptyset$ iff

$aCa^{-1} \cap k_1^{-1}Ck_1 = aCa^{-1} \cap C \neq \emptyset$ it suffices to show that $G(C) \cap A^+$ is compact. Suppose not. Then there exists sequence a_i such that $a_iCa_i^{-1} \cap C \neq \emptyset$ and $\lambda_1^i \to \infty$ where λ_l^i denotes the l^{th} eigenvalue of a_i. Note here that $\lambda_n^i \to 0$ necessarily. Since C is compact the absolute value of every component of a matrix in C is bounded from above by M. Then $(a_m g_p a_m^{-1})_{ij} = \frac{\lambda_i^m}{\lambda_j^m}(g_p)_{ij}$ where $g_p = ((g_p)_{ij}), (g_1, \ldots, g_k) \in C$.

Suppose $\lambda_j^m \to \infty$ as $m \to \infty$ for $j = 1, \ldots, n-1$. Then for $g = (g_1, \ldots, g_k) \in C$ such that $a_m g a_m^{-1} \in C$

$$\left| \frac{\lambda_k^m}{\lambda_n^m}(g_t)_{kn} \right| < M$$

for $k = 1, \ldots, n-1, t = 1, \ldots, k$. Since $\frac{\lambda_k^m}{\lambda_n^m} \to \infty$, $(g_t)_{kn} = 0$ for $k = 1, \ldots, n-1, t = 1, \ldots, k$. This shows that g_t fixes $(0, \ldots, 0, \mathbb{R})$. This is a contradiction to the fact that g is irreducible.

Similarly if $\lambda_j^m \to \infty$ as $m \to \infty$ for $j = 1, \ldots, n-2$ and $\lambda_n^m, \lambda_{n-1}^m$ remain bounded, then for $g \in C$ such that $a_m g a_m^{-1} \in C$, g should fix $(0, \ldots, 0, \mathbb{R}, \mathbb{R})$. This is again a contradiction to the irreducibility. The other cases are similar. This shows that $G(C)$ is compact, so the action is proper. q.e.d.

4. Nonparabolic representations into $\mathrm{PSL}(n, \mathbb{R})$

Let $\Gamma = \pi_1(M)$ be fixed and S be a generating set of it. Denote X a symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ for $n \geq 3$ and d a standard metric on it. For a representation $\rho : \Gamma \to \mathrm{PSL}(n, \mathbb{R})$, one defines

$$d_\rho : X \to \mathbb{R}^+$$

$$x \to \sup_{s \in S} d(x, \rho(s)x).$$

Since the distance function is convex, $d_\gamma(x) = d(x, \gamma x)$ is a convex function, so is d_ρ. One defines the minimum translation length of ρ by

$$\mu(\rho) = \inf_{x \in X} d_\rho(x).$$

It immediately follows from the definition of μ that if $\rho_i \to \rho$ then $\limsup \mu(\rho_i) \leq \mu(\rho)$. The reason is that since $d_{\rho_i} \to d_\rho$ uniformly on any compact set C, for a given $\epsilon > 0$, there exists N such that

$$d_{\rho_i}(x) \leq d_\rho(x) + \epsilon, \ i \geq N, x \in C.$$

So $\inf_{x\in C} d_{\rho_i}(x) \le \inf_{x\in C} d_\rho(x) + \epsilon, i \ge N$. Since this is true for any compact set C, it follows that

$$\limsup \mu(\rho_i) \le \mu(\rho).$$

If one sets $\mathrm{Min}_\rho = \{x \in X | d_\rho(x) = \mu(\rho)\}$, then it is a closed convex set of X. The ideal boundary ∂X is the set of equivalence classes of geodesic rays under the equivalence relation that two rays are equivalent if they are within finite Hausdorff distance each other.

Let γ be isometry acting on X.

(1) We say γ is elliptic if γ has a fixed point in X.

(2) We say γ is hyperbolic if $d_\gamma(x) = d(x, \gamma x)$ assumes the infimum and $l(\gamma) = \inf_{x\in X} d_\gamma(x) > 0$. In this case γ has an invariant geodesic along which it translates.

(3) We say γ is parabolic if d_γ does not assume the infimum $l(\gamma)$. Specially if the infimum is positive, γ is called mixed parabolic. In this parabolic case γ fixes a point x in ∂X and leaves invariant a horosphere based at x. If $x_i \to x$ along a geodesic, then $d_\gamma(x_i) > l(\gamma)$ and $d_\gamma(x_i) \to l(\gamma)$.

See [1, 35] for the details.

We begin with some basic facts about $\mathrm{SL}(n,\mathbb{R})$. The symmetric space X can be identified with the set of positive definite symmetric matrices with determinant 1 and $\mathrm{SL}(n,\mathbb{R})$ acts on X by conjugation $x \to gxg^t$. The isotropy group of I is $\mathrm{SO}(n)$. So $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Fix a base point $I = I \cdot \mathrm{SO}(n)$ in X once and for all. Then the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G = \mathrm{SL}(n,\mathbb{R})$ is

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

where $\mathfrak{k}$ is skew-symmetric matrices and $\mathfrak{p}$ is symmetric matrices with trace 0. The Cartan involution is $Y \to -Y^t$. $\langle Y, Z\rangle = \mathrm{trace}(YZ^t)$ is a positive definite inner product on $\mathfrak{g}$ which is a usual inner product on $\mathbb{R}^{n^2}$. We want to relate the canonical action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ with parabolic subgroups of $\mathrm{SL}(n,\mathbb{R})$.

Any point $y \in \partial X$ is realized as $\gamma_Y(\infty)$ where γ_Y is a unit speed geodesic such that $\gamma_Y(0) = I \cdot \mathrm{SO}(n)$, $\gamma_Y'(0) = Y$ and $Y \in \mathfrak{p}$ with $|Y| = 1$. In other words, the unit tangent bundle $T^1_I(X)$ is identified with ∂X. Let $\lambda_1(Y) > \cdots > \lambda_k(Y)$ be the distinct eigenvalues of Y

and $E_i(Y)$ be the eigenspace of Y in $\mathbb{R}^n$ corresponding to $\lambda_i(Y)$. Set $V_i(Y) = \oplus_{j=1}^i E_j(Y)$. We obtain a flag

$$V_1(Y) \subset \cdots \subset V_k(Y) = \mathbb{R}^n.$$

If m_i denotes the dimension of $E_i(Y)$, since Y is traceless and has norm 1, the following two conditions are satisfied:

(1) $\sum_{i=1}^k m_i \lambda_i(Y) = 0$;

(2) $\sum_{i=1}^k m_i \lambda_i(Y)^2 = 1$.

Then it is easy to see that ∂X can be identified with the set of flags $V_1(Y) \subset \cdots \subset V_k(Y) = \mathbb{R}^n$ such that $\lambda_1(Y) > \cdots > \lambda_k(Y)$ and the two conditions above are satisfied.

If $F = (V_1, \dots, V_k)$ is a flag, then $g \in \mathrm{SL}(n, \mathbb{R})$ acts on F by $gF = (gV_1, \dots, gV_k)$. For $g \in \mathrm{SL}(n, \mathbb{R})$, $(\lambda_1(Y), \dots, \lambda_k(Y)), (V_1(Y), \dots,$ $V_k(Y)$ an eigenvalue-flag pair of some point $y \in \partial X$, it is not difficult to see that $\lambda_i(gy) = \lambda_i(y)$ and $F(gy) = g(F(y))$. See for example [20]. So $gy = y$ iff $g(F(y)) = F(y)$. A subgroup $H \subset \mathrm{SL}(n, \mathbb{R})$ is called parabolic if $Hy = y$ for some $y \in \partial X$. Then we have:

Corollary 1. *A subgroup $H \subset \mathrm{SL}(n, \mathbb{R})$ is parabolic iff the action of H on $\mathbb{R}^n$ is reducible, i.e., it leaves invariant a proper subspace.*

Proof. If $Hy = y$, H fixes a flag $V_1(Y) \subset \cdots \subset V_k(Y) = \mathbb{R}^n$. Since $k \geq 2$, H leaves invariant $V_1(Y)$, so the action of H on $\mathbb{R}^n$ is reducible. Conversely if the action of H is reducible, there exists a proper subspace V_1 of $\mathbb{R}^n$ which is not 0 and left invariant by H. Then one can find $y \in \partial X$ such that $(\lambda(y), F(y))$ is an eigenvalue-flag pair corresponding to $V_1 \subset \mathbb{R}^n$ for some $\lambda(y)$. Then any element in H fixes $F(y)$, so H fixes y. q.e.d.

Next we prove some easy lemmas about the Iwasawa decomposition of $\mathrm{SL}(n, \mathbb{R})$.

Lemma 2. *In a Iwasawa decomposition $\mathrm{SL}(n, \mathbb{R}) = KAN$ where K is an isotropy group of x_0, Ax_0 is a maximal flat and N fixes a Weyl chamber W of Ax_0 at infinity, let M be a subgroup of K which fixes W (so fixes a maximal flat Ax_0 pointwise). Then MA is an abelian subgroup.*

Proof. Let $\mathrm{SL}(n, \mathbb{R})$ act on the set X of positive definite symmetric matrices with determinant 1 by conjugation $x \to gxg^t$. Take a standard

Iwasawa decomposition KAN where K is the isotropy group of the identity matrix I, $\mathrm{SO}(n)$, A is the set of diagonal matrices whose entries are positive and whose determinant is 1, N is the set of upper triangular matrices with 1's on the diagonal. Since M fixes AI pointwise, $YaY^t = a$ for all $a \in A$ and $Y \in M$. So Y must be a diagonal matrix whose entries are ± 1. Then MA is an abelian group. In general, a Iwasawa decomposition as in the statement of the lemma is a conjugate of this standard Iwasawa decomposition. See for example 2.17.27 and 2.17.23 of [20]. This finishes the proof. q.e.d.

Two complete totally geodesic subsets Y_1, Y_2 in a Hadamard manifold are called parallel if their Hausdorff distance is finite, i.e., there exists $N > 0$ such that $d(p_1, Y_2) \le N$ for all points $p_1 \in Y_1$ and $d(p_2, Y_1) \le N$ for all points $p_2 \in Y_2$. In a real analytic Hadamard manifold Z, if M is a complete totally geodesic subset of Z, then the union of all totally geodesic submanifolds parallel to M is isometric to $M \times N$ where N is a closed convex complete subset of Z (Lemma 2.4 of [1]). In our case it is particularly interesting when M is a singular geodesic.

Lemma 3. *If l is a singular geodesic of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ whose infinity point $l(\infty)$ corresponds to diag$(\lambda, \dots, \lambda, \lambda_1, \dots, \lambda_k)$ in $\mathfrak{p}$, then the union of parallels to l is isometric to $\mathbb{R}^k \times \mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k)$.*

Proof. For a point $y \in \partial X$, let $Y \in \mathfrak{p}$ correspond to y, i.e., $\gamma(t) = e^{tY} I \cdot \mathrm{SO}(n)$ is a geodesic starting from $x_0 = I \cdot \mathrm{SO}(n)$ and $\gamma(\infty) = y$. Let $Z_y = \{g \in G \,|\, \mathrm{Ad}(g)Y = Y\} = \{g \in G \,|\, ge^{tY} = e^{tY}g \text{ for all } t\}$. Then for $g \in Z_y$,

$$d(\gamma(t), g\gamma(t)) = d(e^{tY}x_0, ge^{tY}x_0) = d(e^{tY}x_0, e^{tY}gx_0) = d(x_0, gx_0).$$

So $\gamma(t)$ and $g\gamma(t)$ are parallel for all $g \in Z_y$. This shows that $Z_y x_0$ is the union of parallels to $\gamma(t) = l$.

By conjugation, we may assume that $Y = \mathrm{diag}(\lambda, \dots, \lambda, \lambda_1, \dots, \lambda_k)$, i.e., $\lambda_1 = \dots = \lambda_{n-k} = \lambda$. This represents a singular geodesic. Then for $g \in Z_y$, $\mathrm{Ad}(g)Y = Y$ means that $gYg^{-1} = Y$, so $g_{ij} = (YgY^{-1})_{ij}$. A direct calculation shows that

$$(YgY^{-1})_{ij} = \frac{\lambda_i}{\lambda_j} g_{ij}.$$

So to get $\frac{\lambda_i}{\lambda_j} g_{ij} = g_{ij}$, one should have $g_{ij} = 0$ if $\lambda_i \ne \lambda_j$. We get Z_y

equal to

$$\begin{bmatrix} M & 0 \\ 0 & \mu \end{bmatrix}$$

where M are $(n-k)\times(n-k)$ matrices and μ is a diagonal $k\times k$ matrix so that $\det M \times \det\mu = 1$. When $\mu = I$, M is $\mathrm{SL}(n-k,\mathbb{R})$. This shows that the union of parallels to a singular geodesic is isometric to $\mathbb{R}^k \times \mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k)$. q.e.d.

This set which is the union of parallels to a singular geodesic will appear in the proof of the next theorem. We say that a representation is (non)parabolic if its image lies in a (non)parabolic subgroup.

Lemma 4. *A representation ρ is nonparabolic if and only if* Min_ρ *is a nonempty compact set. If $\rho(\Gamma)$ is a discrete, parabolic subgroup with* $\mathrm{Min}_\rho \neq \emptyset$*, then $\rho(\Gamma)$ fixes two end points of some geodeisc l in X.*

Proof. Suppose $\mathrm{Min}_\rho = \emptyset$. Then there exists a sequence $\{x_i\}$ so that $x_i \to x \in \partial X$ and $d_\rho(x_i) \to \mu(\rho)$. This implies that $d(\rho(s)x_i, x_i) \leq \mu(\rho)+\epsilon$ for all large i and $s \in S$. So each $\rho(s)$ fixes x and so does the group generated by S. Then $\rho(\Gamma)$ is parabolic, which is a contradiction.

Now suppose Min_ρ is unbounded. Then there exists $x_i \in \mathrm{Min}_\rho$ such that $x_i \to x \in \partial X$. Now $d(\rho(s)x_i, x_i) = \mu(\rho)$ for all $s \in S$. So $\rho(\Gamma)$ fixes x by the same reasoning. The converse is similar.

For the second statement, let x be an ideal fixed point of $\rho(\Gamma)$. Since ρ is parabolic and by the assumption, Min_ρ is noncompact. Take a geodesic l emanating from x and $L = \mathrm{Min}_\rho \cap l$ noncompact. Choose $x_0 \in L$ and take a generalized Iwasawa decomposition $G_x = NAK$, see Proposition 2.17.5 (4) in [20]. Here G_x is a parabolic subgroup fixing x, K is an isotropy subgroup of x_0 (actually, it fixes l pointwise), Ax_0 is the union of parallels to l, and N is the horospherical subgroup which is determined only by x. Note for any $n \in N$, nl and l are never parallel. Furthermore $d(ny, y) \to 0$ as $y \in l'$ goes to x where l' is a geodesic emanating from x. Let S be a fixed generating set of Γ as before. For any $g \in S$, $g = nak$. If $n \neq id$, as $y \in L$ tends to x, $d(nak(y), y) = d(na(y), y)$ strictly decreases since al emanates from x. This is a contradiction to the definition of Min_ρ. So any element in S sends l to a parallel geodesic, particularly it fixes two end points of l. So $\rho(\Gamma)$ fixes two end points of l. q.e.d.

We will need the following theorem later in Section 10. The reader may skip this theorem until Section 10.

Theorem 1. *Suppose* Γ *is centerless, not solvable and suppose* $\rho_k \to \tau$ *in* $R = \mathrm{Hom}(\Gamma, \mathrm{PSL}(n, \mathbb{R}))$ *where* ρ_k *are nonparabolic discrete faithful representations with Zariski dense images either in* $\mathrm{PSL}(n, \mathbb{R})$ *or in* $\mathrm{PSO}(n-1, 1)$*. Suppose* ρ_k *have lifts* $\widetilde{\rho_k}$ *to* $\mathrm{SL}(n, \mathbb{R})$*. Then after passing to a finite index subgroup of* Γ *and conjugating* ρ_k*, there exists a discrete faithful representation* ρ *such that either* ρ *is nonparabolic into* $\mathrm{PSL}(n, \mathbb{R})$ *or parabolic into* $\mathrm{SL}(n, \mathbb{R})$ *acting on* $\mathrm{SL}(n-k, \mathbb{R})/\mathrm{SO}(n-k), k = 1, \dots, n-2$ *as a discrete faithful group. In either case,* ρ_k*(resp.* $\widetilde{\rho_k}$*)*$\to \rho$ *and* $g_k \rho g_k^{-1} \to \tau$*(resp.* $\widetilde{\tau}$*) for some sequence* $\{g_k\}$*.*

Proof. Firstly, by [26], τ is a discrete faithful representation. More precisely, since the image $\rho_k(\Gamma)$ is Zariski dense in a centerless semisimple Lie group, $\rho_k(\Gamma)$ has no nontrivial nilpotent normal subgroups (Lemma 1.2 of [26]). By Selberg's lemma, there exists a finite index subgroup Γ' so that $\rho_k(\Gamma')$ has no torsion. Then $\tau(\Gamma')$ is discrete and faithful by the Lemma 1.1 of [26]. Since Γ' is of finite index, $\tau(\Gamma)$ is still discrete. If $Z \subset \Gamma$ is a kernel of τ, $\rho_k(Z)$ is a normal discrete subgroup of $\rho_k(\Gamma)$. Since $\rho_k(\Gamma)$ is Zariski dense in a simple group, $\rho_k(Z)$ is a normal discrete subgroup of that simple group, so included in a center. But since $\mathrm{PSL}(n, \mathbb{R})$ (or $\mathrm{PSO}(n-1, 1)$) has no center, it is trivial. This shows that τ is faithful. By conjugating if necessary, we can asssume that $x_0 \in \mathrm{Min}_{\rho_k}$ for all k. We have $\limsup\{\mu(\rho_k) = d_{\rho_k}(x_0)\} \leq \mu(\tau)$. Then since $d_{\rho_k}(x_0) \leq \mu(\tau) + \epsilon$ for large k, $\rho_k(S)x_0$ is contained in a compact set of X, so one can extract a subsequence which converges to ρ. After passing to a subsequence, we assume $\rho_k \to \rho$. By the same argument above ρ is discrete and faithful. If we use $\widetilde{\rho_k} \to \widetilde{\rho}$ in $\mathrm{SL}(n, \mathbb{R})$, since $\widetilde{\rho_k}$ covers ρ_k, $\widetilde{\rho_k}$ is discrete and faithful and $\widetilde{\rho}$ covers ρ. So $\widetilde{\rho}$ is also discrete and faithful.

Suppose ρ is parabolic, then Min_ρ is either empty or unbounded. In this case we use $\widetilde{\rho_k} \to \widetilde{\rho}$ in $\mathrm{SL}(n, \mathbb{R})$. Abusing notations, we will use the same notations ρ_k, ρ for $\widetilde{\rho_k}, \widetilde{\rho}$. Since convex functions d_{ρ_k} converge to the convex function d_ρ uniformly, $x_0 \in \mathrm{Min}_\rho$. So Min_ρ is unbounded. Theny by Lemma 4, ρ should fix end points of some geodesic l.

Take a set W which is the union of all parallels to l. Then W is isometric to $l \times Y$ where Y is a closed convex complete subset of X, see [1] Lemma 2.4 (if l is nonsingular, W is a unique maximal flat containing l). Furthermore W is $\rho(\Gamma)$ invariant. Take a Iwasawa decomposition KAN where K is an isotropy group of x_0, Ax_0 is a maximal flat containing l and N is a Nilpotent group fixing a Weyl chamber at infinity containing $l(\infty)$.

We divide the situation into two cases. First suppose l is nonsingular, then W is a unique maximal flat containing l. Since $\rho(\Gamma)$ leaves W invariant, $\rho(\Gamma) \subset MA$ where M is a subgroup of K fixing $l(\pm\infty)$. Then MA is an abelian group by Lemma 2. So $\rho(\Gamma)$ is an abelian group, which is a contradiction.

Now suppose l is singular. By Lemma 3,

$$W = \mathbb{R}^k \times \mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k)$$

if the singular direction is $\mathrm{diag}(\lambda, \dots, \lambda, \lambda_1, \dots, \lambda_k)$ and it is a convex simply connected totally geodesic subset of X (see Proposition 2.11.4 of [20]). Therefore $\rho(\Gamma)$ can be conjugate to a subgroup of the form

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_k \end{bmatrix}$$

where M is a $(n-k) \times (n-k)$ matrix.

Since $\rho(\Gamma)W = W$ and W is a symmetric space $\mathbb{R}^k \times \mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k)$, we can think of $\rho(\Gamma)$ as an isometry group acting on W. Since $\mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k)$ has no Euclidean de Rham factor, $\rho(\Gamma)$ preserves the splitting $\mathbb{R}^k \times \mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k)$. Let $P : \rho(\Gamma) \to \mathrm{Iso}(\mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k))$ be a projection. Then by Proposition 7.2.2 of [20], $P(\rho(\Gamma))$ is either discrete or solvable.

Since $\mathrm{Iso}(\mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k))$ has a finite number of component, by taking a finite index subgroup of Γ, we can assume that $P : \rho(\Gamma) \to \mathrm{SL}(n-k,\mathbb{R})$. In this case explicitly $P(g) = \frac{1}{(detM)^{1/n-k}} M$ for $g \in \rho(\Gamma)$ of the form

$$\begin{bmatrix} M & 0 \\ 0 & \mu \end{bmatrix}.$$

Suppose P has a kernel. Then any element in the kernel is of the form

$$\begin{bmatrix} cI & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_k \end{bmatrix}$$

so it is central in $\rho(\Gamma)$. Now since $\rho(\Gamma)$ is discrete and faithful, Γ has a center, which is a contradiction. So $P\rho$ is faithful.

If $P\rho(\Gamma)$ is discrete, we have a discrete and faithful representation $P\rho$ of Γ into $\mathrm{SL}(n-k,\mathbb{R})$ acting on $\mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k)$ after passing to a finite index subgroup of Γ. If $P\rho(\Gamma)$ is solvable, Γ is solvable, which is a contradiction.

So ρ is either a nonparabolic discrete faithful representation into $\mathrm{PSL}(n,\mathbb{R})$ or a discrete faithful parabolic representation into $\mathrm{SL}(n,\mathbb{R})$ acting on $\mathrm{SL}(n-k,\mathbb{R})/\mathrm{SO}(n-k)$ as a discrete faithful group after passing to a finite index subgroup of Γ. By the construction of ρ it is obvious that $\rho_k \rightarrow \rho$ and $g_k\rho_k g_k^{-1} \rightarrow \tau$ for some $\{g_k\}$. This implies that $g_k\rho g_k^{-1} \rightarrow \tau$. This finishes the proof. q.e.d.

Since $\mathrm{PGL}(n,\mathbb{R}) = \mathrm{PSL}_{-}(n,\mathbb{R}) \cup \mathrm{PSL}(n,\mathbb{R})$ (see Section 3), note that any holonomy representation of a convex real projective structure can be lifted to $\mathrm{PSL}(n,\mathbb{R})$ up to the index two subgroup of $\pi_1(M)$, i.e, there exists an index two subgroup Γ of $\pi_1(M)$ so that the restriction of the holonomy representation to Γ has an image in $\mathrm{PSL}(n,\mathbb{R})$. The following lemma shows that it can be lifted to $\mathrm{SL}(n,\mathbb{R})$.

Lemma 5. *If $\rho : \pi_1(M) \rightarrow \mathrm{PSL}(n+1,\mathbb{R})$ is a holonomy representation of a strictly convex real projective structure of a closed manifold M, then it lifts to a representation $\widetilde{\rho}$ into $\mathrm{SL}(n+1,\mathbb{R})$ when $n+1$ is even.*

Proof. Let $M = \Omega/\rho(\pi_1(M))$ where Ω is a convex domain in $\mathbb{RP}^n$. Let C be a component of two lifts of Ω in S^n. Let $\gamma_1, \dots, \gamma_k$ be a generator of $\pi_1(M)$. Choose a matrix A_i out of $\rho(\gamma_i) = \pm A_i$ so that $A_iC = C$. Then clearly the group generated by A_i preserves C. If γ is a word $W(\gamma_i)$, set $\widetilde{\rho}(\gamma) = W(A_i)$. Suppose $R(\gamma_i)$ is a relation of $\pi_1(M)$ with respect to the generators γ_i. Then $\rho(R) = \pm I$. If $\widetilde{\rho}(R) = -I$, $\widetilde{\rho}(R) = R(A_i)$ does not preserve C, which is impossible. So $\widetilde{\rho}(R) = I$. This shows that $\widetilde{\rho}$ is a lift of ρ. q.e.d.

Due to this lemma we can think of a holonomy representation of a convex real projective structure as a representation either in $\mathrm{PSL}(n,\mathbb{R})$ or in $\mathrm{SL}(n,\mathbb{R})$ up to index two subgroup.

5. Openess of convex real projective structures

In this section we want to show that the set $\mathfrak{B}(M)$ of strictly convex real projective structures on M is open in the set of all real projective structures $\mathbb{RP}^n(M)$ on a closed manifold M. In [38], Koszul showed that the space of affine structures on M whose developing image is a convex set containing no complete straight line (he called it hyperbolic) is open in the space of affine structures on M. Later Vey [44] showed that such a structure has a cone as the developing image on which a group of affine transformations acts cocompactly and properly. But Benzécri [6] showed that every $\mathbb{RP}^n$ manifold M has a naturally associated flat affine manifold $M \times S^1$. By the construction, a strictly convex real projective structure on M induces a convex affine structure on $M \times S^1$ whose developing image is a cone. Since such affine structures on $M \times S^1$ is open, $\mathfrak{B}(M)$ is open in $\mathbb{RP}^n(M)$.

6. Proximality and invariant convex cones

An element $g \in \mathrm{GL}(n, \mathbb{R})$ is called proximal if $\lambda_1(g) > \lambda_2(g)$ where $\lambda_1(g) \geq \lambda_2(g) \geq \cdots \geq \lambda_n(g)$ is the sequence of modules of eigenvalues of g repeated with multiplicity. In this case the eigenvalue corresponding to $\lambda_1(g)$ is real. Equivalently an element is proximal if it has an attracting fixed point in $\mathbb{RP}^{n-1}$. It is biproximal if g^{-1} is also proximal. A proximal element is called positively proximal if the eigenvalue corresponding to $\lambda_1(g)$ is a positive real number. An element $g \in \mathrm{GL}(n, \mathbb{R})$ is called positively biproximal if it is biproximal and, the eigenvalue corresponding to $\lambda_1(g)$ and the eigenvalue corresponding to $\lambda_n(g)$ have the same sign. It is easy to see that a biproximal element leaves invariant a convex cone with nonempty interior in $\mathbb{RP}^{n-1}$ iff g is positively biproximal. See the remark after Lemma 4.5 of [3]. One says that $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$ is proximal if it contains a proximal element and positively proximal if every proximal element in Γ is positively proximal. One says that a discrete subgroup Γ divides a bounded convex cone $C \subset \mathbb{R}^n$ if it preserves C and C/Γ is compact. The following theorem is due to [3] (Proposition 1.1 and Theorem 3.6).

Theorem A. *If a discrete subgroup $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ divides an open, bounded, strictly convex cone $C \subset \mathbb{RP}^{n-1}$ which is not an ellipsoid (in this case C is a real hyperbolic space), then Γ is Zariski dense in $\mathrm{SL}(n, \mathbb{R})$. When C is an ellipsoid, Γ is Zariski dense in $\mathrm{SO}(n-1, 1)$.*

When M is a closed convex real projective surface with $\chi(M) < 0$, Kuiper [39, 6, 32] showed that:

(1) C is strictly convex and ∂C is at least C^1. Either ∂C is a conic in $\mathbb{RP}^2$ or is not $C^{1+\epsilon}$ for any $0 < \epsilon < 1$.

(2) If $A \in \Gamma$ is nontrivial, A is positively biproximal. Every homotopically nontrivial closed curve on M is freely homotopic to a unique closed geodesic in the Hilbert metric.

7. length of a closed geodesic

In this section we calculate the length of a closed geodesic in terms of eigenvalues of the corresponding element in the group. Let $M = C/\Gamma$ be a closed strictly convex real projective manifold where $\Gamma \subset \mathrm{PGL}(n, \mathbb{R}) = \mathrm{Aut}(\mathbb{RP}^{n-1})$. If l is a closed geodesic in M, then its lift $\widetilde{l}$ to C is an invariant geodesic of some element $g \in \Gamma$ on which g acts as a translation with the translation length $l(g)$. One end point of $\widetilde{l}$ is a repelling fixed point of g corresponding to $\lambda_n(g)$ and the other end point is an attracting fixed point corresponding to $\lambda_1(g)$. Since the attracting fixed point of g^{-1} is equal to the repelling fixed point of g, it follows that g is positively biproximal. So the attracting fixed point in $\mathbb{PR}^{n-1}$ is the one dimensional eigenspace in $\mathbb{R}^n$ corresponding to $\lambda_1(g)$ and the repelling fixed point is the eigenspace corresponding to $\lambda_n(g)$. Conjugating g, g is of the form

$$\begin{bmatrix} \pm\lambda_1(g) & * & 0 \\ 0 & \cdots & 0 \\ 0 & * & \pm\lambda_n(g) \end{bmatrix}$$

Then the attracting fixed point is $[1, \ldots, 0]$ ($= \infty$ on the extended real line) and the repelling fixed point is $[0, \ldots, 1]$ (=0 on the extended real line) in homogeneous coordinates. The invariant geodesic of g joining these two points is $[s, 0, \ldots, 1-s] = \frac{s}{1-s}, 0 \le s \le 1$.

Proposition 1. *If g corresponds to a closed geodesic l in M, then the length of l is $l(g) = \log \lambda_1(g) - \log \lambda_n(g)$.*

Proof. Since $l(g) = d(x, gx)$ for any x on the invariant geodesic of g,

$$\begin{aligned} l(g) &= d([s, 0, \ldots, 1-s], [\pm\lambda_1(g)s, \ldots, \pm\lambda_n(g)(1-s)]) \\ &= \log \frac{\left|\infty - \frac{s}{1-s}\right| \left|\frac{\lambda_1(g)s}{\lambda_n(g)(1-s)}\right|}{\left|\infty - \frac{\lambda_1(g)s}{\lambda_n(g)(1-s)}\right| \left|\frac{s}{1-s}\right|} = \log \lambda_1(g) - \log \lambda_n(g). \end{aligned}$$

q.e.d.

8. Marked length rigidity in Hilbert metric

In this section we prove the main theorem. We begin with a proposition. Suppose $M = C_1/\Gamma_1, N = C_2/\Gamma_2$ are compact, strictly convex real projective manifolds with the same marked length spectrum, i.e., there exists an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that $l(g) = l(\phi(g))$ for all $g \in \Gamma_1$. Our goal is to show that Γ_1 and Γ_2 are conjugate.

Proposition 2. *Suppose Γ_1 and Γ_2 have the same marked length spectrum. Let G_i be the Zariski closure of Γ_i. It is either $\mathrm{PSL}(n,\mathbb{R})$ or $\mathrm{PSO}(n-1,1)$ depending whether the projective manifold is hyperbolic or not. Then the graph group $\Gamma = \{(g,\phi(g)) \mid g \in \Gamma_1\} \subset G_1 \times G_2$ is not Zariski dense in $G_1 \times G_2$.*

Proof. Let A_i be a fixed maximal abelian group in the Iwasawa decomposition $G_i = K_i A_i N_i$. Explicitly A_i can be taken as the set of diagonal matrices. Let A_i^+ be a fixed Weyl chamber, explicitly A_i^+ can be taken as the set of diagonal matrices $(a_1, \dots, a_n)$ such that $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ if $G_i = \mathrm{PSL}(n,\mathbb{R})$, as the set of diagonal matrices $(\lambda, 1, \dots, 1, \frac{1}{\lambda}), \lambda \geq 1$ when $G_i = \mathrm{PSO}(n-1,1)$. Denote $\mathfrak{g}_i, \mathfrak{a}_i, \mathfrak{a}_i^+$ Lie algebra of G_i, maximal abelian Lie subalgebra, Weyl chamber correspondingly. Any element $g \in G_i$ has a unique Jordan decomposition $g = ehu$ where e is elliptic, h hyperbolic and u unipotent. Then h is conjugate to a unique element a in A_i^+. Let

$$\lambda : G_i \to \mathfrak{a}_i^+$$

be a map defined by $e^{\lambda(g)} = a$. In our case, $\lambda(g)$ is a vector in $\mathfrak{a}_i^+$ whose coordinates are logarithms of the absolute values of eigenvalues of g in a decreasing order. Now we appeal to Benoist's Theorem [4] that if Γ is a Zariski dense subgroup of a semisimple Lie group G, then the limit cone which is the closure of the image $\lambda(\Gamma)$ has nonempty interior in $\mathfrak{a}$.

A Cartan subalgebra of $G_1 \times G_2$ is $\mathfrak{a}_1 \times \mathfrak{a}_2$. But Γ has the property that

$$\log \lambda_1(g) - \log \lambda_n(g) = \log \lambda_1(\phi(g)) - \log \lambda_n(\phi(g)),$$

so its limit cone is contained in the closed subset of $\mathfrak{a}_1 \times \mathfrak{a}_2$

$$\{[(v_1, \dots, v_n), (w_1, \dots, w_n)] \mid v_1 - v_n = w_1 - w_n\}.$$

But this set has empty interior in $\mathfrak{a}_1 \times \mathfrak{a}_2$. So Γ is not Zariski dense.

q.e.d.

Now we prove our main theorem.

Theorem 2. *Let M and N be compact, strictly convex real projective manifolds with Hilbert metrics. If they have the same marked length spectrum then they are projectively equivalent.*

Proof. Set $M = C_1/\Gamma_1, N = C_2/\Gamma_2$ where C_i is a bounded strictly convex cone and $\Gamma_i \subset G_i$ where $G_i = \mathrm{PSL}(n, \mathbb{R})$ or $\mathrm{PSO}(n-1, 1)$ depending whether the manifold is hyperbolic or not. By Theorem A, Γ_i is Zariski dense in G_i. Let $\phi : \Gamma_1 \to \Gamma_2$ be an isomorphism preserving the translation lengths. Then by the previous proposition, $\Gamma = \{(g, \phi(g)) | g \in \Gamma_1\}$ is not Zariski dense in $G_1 \times G_2$. Denote G the Zariski closure of Γ in $G_1 \times G_2$. Then $G \neq G_1 \times G_2$ by Proposition 2. Let P_i be the projection from G onto each factor G_i. Since Γ normalizes G, Γ_i normalizes $P_i(G)$. For Γ_i is Zariski dense, $P_i(G) = G_i$ since G_i is a simple Lie group. This shows that P_i is surjective.

Next goal is to show that P_i is injective. Consider $ker P_1$. If its Lie algebra is trivial, it is a discrete normal subgroup of $\{e\} \times G_2 = G_2$, so it is included in the center of G_2. Since G_2 is centerless, $ker P_1$ is trivial. If its Lie algebra is not trivial, since $ker P_1$ is normal in $\{e\} \times G_2 = G_2$, it must be $\{e\} \times G_2$ since G_2 is simple. This fact and $P_1(G) = G_1$ would imply that $G = G_1 \times G_2$, which is a contradiction to the fact that $G \neq G_1 \times G_2$.

This shows that $ker P_1$ is trivial. Similarly $ker P_2$ is trivial. So P_i is an isomorphism. Then $\rho = P_2 \circ P_1^{-1}$ is a continuous isomorphism from G_1 to G_2 which coincides with ϕ on Γ_1.

Then it is a standard fact that if $\rho : G_1 \to G_2$ is a continuous isomorphism between two semisimple Lie groups G_1 and G_2 extending an isomorphism ϕ, then ρ induces an isometry F between two symmetric spaces G_1/K_1 and G_2/K_2 (where K_i is a maximal compact subgroup) and ϕ is a conjugation by F. In our case, G_i is a simple Lie group, so Γ_1 and Γ_2 are conjugate by an isometry F in $\mathrm{SL}(n, \mathbb{R}) \cup \mathrm{SL}_{-}(n, \mathbb{R})$. This shows that M and N are isometric with respect to the Hilbert metric and projectively equivalent. q.e.d.

9. Cross-ratio on the ideal boundary of a cone

Bourdon [9] proved that a cross-ratio preserving homeomorphism from the ideal boundary of a rank one symmetric space to the ideal boundary of a $\mathrm{CAT}(-1)$ space can be extended isometrically to the whole spaces. Using this fact, the author [34] proved that if a compact quotient of a $\mathrm{CAT}(-1)$ space and a compact negatively curved locally symmetric manifold have the same marked length spectrum, then they are isometric. The main idea there was to prove that the marked length spectrum determines the cross-ratio on the ideal boundary of CAT(-1) space. In this section, we show the same thing for compact, strictly convex real projective manifolds.

Theorem 3. *Let $M = C_1/\Gamma_1$ and $N = C_2/\Gamma_2$ be compact strictly convex real projective manifolds. Then there is a cross-ratio preserving equivariant homeomorphism between ∂C_1 and ∂C_2 iff M and N are projectively equivalent.*

To prove this theorem we introduce a rigorous definition of the cross-ratio and several preliminary lemmas.

Let $\gamma(t)$ be a geodesic ray. Let x be fixed. Then the function

$$t \mapsto d(x, \gamma(t)) - t$$

is monotone decreasing since for $s < t$,

$$d(x, \gamma(t)) \leq d(x, \gamma(s)) + t - s$$

by triangle inequality. Define the Busemann function by

$$h_\gamma(x) = \lim_{t\to\infty} [d(x, \gamma(t)) - t].$$

Each level set of h_γ is called the horosphere. Let $\beta(t)$ be a geodesic with $\beta(\infty) = \gamma(\infty)$. If $a = \beta \cap h_\gamma^{-1}(t), b = \beta \cap h_\gamma^{-1}(s)$, then $d(a, b) = |t - s|$. Let X be a metric space such that given four distinct points on the ideal boundary, there are four disjoint horospheres based at each point and there exists a unique geodesic connecting two points on the ideal boundary.

Definition 4. The cross-ratio of four point x_1, x_2, x_3, x_4 on the ideal boundary of X is defined as follows. Let l_{ij} be a unique geodesic connecting x_i and x_j. Let H_i be the four disjoint horospheres based at x_i. Let s_{ij} be the geodesic segment cut out by the horospheres H_i, H_j. Then the cross-ratio of four points is defined by

$$[x_1, x_2, x_3, x_4] = l(s_{13}) + l(s_{24}) - l(s_{14}) - l(s_{23})$$

where $l(s_{ij})$ is the length of the segment.

Note that the definition is independent of the choice of horospheres by the property of horospheres.

Suppose X has the property that if two geodesics l_1, l_2 share the forward end point in the ideal boundary, then $d(l_1 \cap H_t, l_2 \cap H_t) \to 0$ as $t \to \infty$ where H_t are horospheres based at the forward end point of geodesics shrinking to the end point. Then we have the following proposition.

Proposition 3. *Let X be a metric space as in Definition 4. Suppose X has the property that if two geodesics l_1, l_2 share the forward end point in the ideal boundary, then $d(l_1 \cap H_t, l_2 \cap H_t) \to 0$ as $t \to \infty$ where H_t are horospheres based at the forward end point of geodesics shrinking to the end point. Then the cross-ratio of four points x_1, x_2, x_3, x_4 on the ideal boundary can be defined by*

$$[x_1, x_2, x_3, x_4] = \lim_{n\to\infty} \{d(x_1^n, x_3^n) + d(x_2^n, x_4^n) - d(x_1^n, x_4^n) - d(x_2^n, x_3^n)\}$$

where x_0 is a point in X and $x_i^n \to x_i$ and x_i^n lies on the geodesic ray from x_0 to x_i.

Proof. Let H_i^n be the horosphere based at x_i and passing through x_i^n. Then by the assumption, $d(x_i^n, H_i^n \cap l_{ij}) \to 0$ where l_{ij} is a unique geodesic connecting x_i and x_j. So $d(x_i^n, x_j^n) \to l(s_{ij}^n)$ where s_{ij}^n is the geodesic segment on l_{ij} cut out by horospheres H_i^n, H_j^n. Then the claim follows from the definition. q.e.d.

We first prove that the difference between two asymptotic geodesics tends to zero as they approach to the end point in a convex domain equipped with the Hilbert metric. Note that by [5], the boundary of any strictly convex domain which admits a compact quotient is necessarily C^1.

Proposition 4. *Let Ω be a strictly convex domain in $\mathbb{R}^n$ with C^1 boundary which is equipped with the Hilbert metric. Let $l_1(t), l_2(t)$ be two geodesic rays with the same forward end point P. There are sequences $\{t_n\}, \{s_n\}$ tending to the same ideal point P such that*

$$\lim_{n\to\infty} d(l_1(t_n), l_2(s_n)) = 0.$$

Proof. First consider the case when $n = 2$. Since $\partial\Omega$ is C^1, there exists a tangent line l at P.

Thinking of l as the x-axis with origin equal to P, nearby part of $\partial\Omega$ around P is the graph of some convex C^1 function $f(x)$ since $\partial\Omega$ is C^1 and convex. Let l_c be a horizontal line with y coordinate equal to c. Denote the points at which l_c meets with $\partial\Omega, l_1, l_2, \partial\Omega$ in this order by

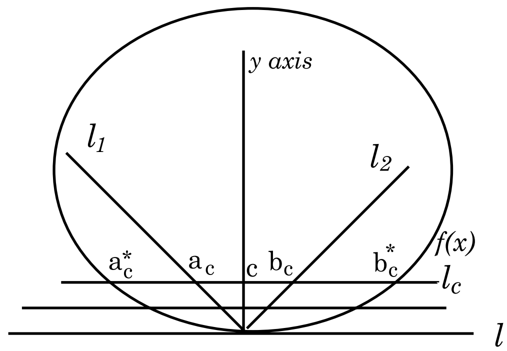

Figure 1: Two geodesic rays getting closer by the distance 0.

a_c^*, a_c, b_c, b_c^*. (See Figure 1.) Let l_1 be the graph of $y = \beta x$, l_2 the graph of $y = \alpha x$. Let the equation of the line connecting the origin and a_c^* be $y = \beta_c x$, and let the equation of the line connecting the origin and b_c^* be $y = \alpha_c x$. Then clearly $\alpha_c, \beta_c \to 0$ as $c \to 0$. Then

$$\begin{aligned} d(a_c, b_c) &= \log \frac{|b_c - a_c^*|\,|a_c - b_c^*|}{|a_c^* - a_c|\,|b_c - b_c^*|} \\ &= \log \frac{\left|\frac{c}{\alpha} - \frac{c}{\beta_c}\right| \left|\frac{c}{\beta} - \frac{c}{\alpha_c}\right|}{\left|\frac{c}{\beta_c} - \frac{c}{\beta}\right| \left|\frac{c}{\alpha} - \frac{c}{\alpha_c}\right|} \\ &= \log \frac{\left|\frac{\beta_c}{\alpha} - 1\right| \left|\frac{\alpha_c}{\beta} - 1\right|}{\left|1 - \frac{\beta_c}{\beta}\right| \left|\frac{\alpha_c}{\alpha} - 1\right|}. \end{aligned}$$

Then as $c \to 0$,$d(a_c, b_c) \to 0$. For the higher dimensional case, we take a tangent hyperplane at P and do the same thing to conclude the claim. We use only the fact that the tangent plane exists at P and that the domain is strictly convex. q.e.d.

We have the following corollary.

Corollary 2. *Let γ, β be two geodesics with $\gamma(\infty) = \beta(\infty)$. Then $d(\gamma \cap H_t, \beta \cap H_t) \to 0$ where H_t is a horosphere shrinking to $\gamma(\infty)$ as $t \to \infty$.*

Proof. Suppose the distance is bounded below by some positive number δ. Parameterize the geodesics so that $\beta(0)$ and $\gamma(0)$ lie on the same horosphere. Then $\gamma(t), \beta(t)$ lie on the same horosphere. So

$d(\beta(t),\gamma(t)) > \delta$. When t is large, by the above proposition, there is a point x_t on β such that $d(x_t,\gamma(t)) \leq \epsilon_t$ and $\lim_{t\to\infty}\epsilon_t = 0$. By the triangle inequality we have $\delta - \epsilon_t < d(\beta(t), x_t) < \delta + \epsilon_t$. On the other hand, we also have

$$d(\gamma(s),\gamma(t)) - \epsilon_t < d(x_t,\gamma(s)) < d(\gamma(s),\gamma(t)) + \epsilon_t$$

for $s \gg t$. So we get

$$-t - \epsilon_t < d(x_t,\gamma(s)) - s < -t + \epsilon_t,$$

which implies that

$$-t - \epsilon_t < h_\gamma(x_t) < -t + \epsilon_t.$$

This contradicts to the fact that $d(x_t,\beta(t)) > \delta - \epsilon_t$ when t is large so that $\delta - \epsilon_t > \epsilon_t$ since $h_\gamma(\beta(t)) = -t$. q.e.d.

With the aid of Proposition 3, one can define the cross-ratio on any four points in $X \cup \partial X$ by

$$\begin{aligned}&[x,y,z,w]\\ &= \lim_{(x_i,y_i,z_i,w_i)\to(x,y,z,w)} \{d(x_i,z_i) + d(y_i,w_i) - d(x_i,w_i) - d(y_i,z_i)\}.\end{aligned}$$

Then $[a,b,x,y] + [b,c,x,y] = [a,c,x,y]$. Let x be a point on the invariant axis of α, then

$$\begin{aligned}[\xi_1,\xi_2,\eta,\alpha(\eta)] &= \sum_{n\in\mathbb{Z}} [\alpha^n(x),\alpha^n(\alpha(x)),\eta,\alpha(\eta))]\\ &= \sum_{n\in\mathbb{Z}} [x,\alpha(x),\alpha^{-n}(\eta),\alpha^{-n}(\alpha(\eta))]\\ &= [x,\alpha(x),\xi_1,\xi_2] = -2l(\alpha)\end{aligned}$$

where η is a point in ∂X, ξ_1 and ξ_2 are the repelling and the attracting fixed points of α. This shows that if we know the cross-ratios on the ideal boundary ∂X, we know the marked length spectrum. See [42, 33].

Conversely if we know the marked length spectrum, then we know the cross-ratio on the ideal boundary [33]. The same proof there applies to the Hilbert metric. See [33, Theorem 1].

Proposition 5. *Let a, b be two biproximal isometries in X equipped with the Hilbert metric. Let a^-, b^- be the repelling fixed points of a and b, a^+, b^+ the attracting fixed points of a and b. Then*

$$\lim_{n\to\infty} l(a^n) + l(b^n) - l(b^n a^n) = [a^-, b^-, a^+, b^+].$$

Since any two points in the limit set can be approximated by the repelling and attracting fixed point of some biproximal element [3] (Lemma 2.5 and 2.6), [4] (Lemma 3.6) and the cross-ratio is a continuous function, the marked length spectrum determines the cross-ratio on the ideal boundary.

Proof of Theorem 3. Suppose there is a cross-ratio preserving equivariant homeomorphism between the boundaries of two strictly convex domains. Then Γ_1 and Γ_2 have the same marked length spectrum since the homeomorphism maps two end points of an invariant geodesic of an element in Γ_1 to the end points of the invariant geodesic of the corresponding element in Γ_2. Then by our main theorem, M and N are projectively equivalent.

Conversely if M and N are projectively equivalent, they have the same marked length spectrum. Let $\phi : \Gamma_1 \rightarrow \Gamma_2$ be an isomorphism inducing projective equivalency and preserving marked length spectrum. Then one can construct a cross-ratio preserving equivariant homeomorphism between the ideal boundaries. Such a map is constructed as follows: first define the map f on the set of attracting fixed points of Γ_1 by sending attracting fixed points of elements in Γ_1 to the attracting fixed points of corresponding elements in Γ_2. Then by Proposition 5, f preserves cross-ratio. Since the set of attracting fixed points is dense in the ideal boundary [3] (Lemma 2.5), for any x in the ideal boundary, there exit attracting fixed points x_i converging to x.

Put $y_i = f(x_i)$. Since the ideal boundary is compact there is a subsequence $\{y_k\}$ of $\{y_i\}$ which converges to y. Define $f(x) = y$. We should check y is the only accumulation point of y_i. Suppose y_l converges to another point z. Fix two distinct points p, q different from x which are attracting fixed points. Set $f(p) = t,\ f(q) = s$. If we put $f(x_k) = y_k,\ f(x_l) = y_l$, then by Proposition 5 we have

$$[p, x_k, x_l, q] = [t, y_k, y_l, s]$$

Since the cross-ratio is a continuous function, the limit of above cross-ratios should be the same. But $[p, x_k, x_l, q]$ tends to ∞ since x_k, x_l converge to x while $[t, y_k, y_l, s]$ tends to $[t, y, z, s]$ which is finite. This is a contradiction.

This way the map can be continuously extended to the whole ideal boundary and by Proposition 5 this map is cross-ratio preserving. See the argument in [33, 34] for details. q.e.d.

10. The structure of the space of convex real projective structures

In this section we want to focus on the space of real projective structures and prove an analogue of a Hitchin's conjecture in dimension 3. First note that $\mathrm{PSL}(n,\mathbb{R})$ acts on $\mathbb{RP}^{n-1}$ as usual and also acts as an isometry group on a symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

Definition 5. A subgroup $\Gamma \subset \mathrm{PSL}(n,\mathbb{R})$(resp., $\mathrm{GL}(n,\mathbb{R})$) is **irreducible** if it does not leave invariant any proper projective subspace (resp., any proper subspace) of $\mathbb{RP}^{n-1}$(resp., $\mathbb{R}^n$).

In [31, Section 5], Johnson and Millson showed that $\mathbb{RP}^n(M)$ has dimension greater than or equal to r where M is a closed real hyperbolic manifold which contains r number of disjointly embedded two-sided connected totally geodesic hypersurfaces. Actually they showed that $\mathbb{RP}^n(M)$ contains an r-ball around the hyperbolic structure on M. As in [25, section 3.7], from openness of the convex structures (see also Section 5 of this article), small deformations of a hyperbolic structure are still convex. This shows that $\mathfrak{B}(M)$ has dimension at least r.

Now we want to show that

$$h : \mathfrak{B}(M) \rightarrow \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R})$$

is injective. Let $\mathfrak{R} = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R})$ and let $l : \mathfrak{R} \rightarrow \mathbb{R}^{\pi_1(M)}$ be defined by $l(\rho) = (\log \lambda_1(\rho(\gamma)) - \log \lambda_n(\rho(\gamma)))_{\gamma \in \pi_1(M)}$. Then by our main theorem, $l \circ h$ is injective on $\mathfrak{B}(M)$, and so is h on $\mathfrak{B}(M)$. From now on we denote by h the map restricted on $\mathfrak{B}^0(M)$ where $\mathfrak{B}^0(M)$ is a component of $\mathfrak{B}(M)$ containing a hyperbolic structure on M. This shows that $h : \mathfrak{B}^0(M) \rightarrow \mathfrak{R}$ is an embedding onto an open set. We just remark that a holonomy representation of a strictly convex real projective structure is discrete, faithful and Zariski dense (Theorem A) in $\mathrm{PSL}(n,\mathbb{R})$.

Let M be a real hyperbolic closed manifold. A component of $\mathfrak{R}$ which contains the deformation space of hyperbolic structures on M is called the Teichmüller component. Note that the deformation space of hyperbolic structures on M is a point if the dimension of M is > 2. The following theorem shows that $h(\mathfrak{B}^0(M))$ is exactly equal to the Teichmüller component. Such a theorem is known for surfaces [14, 15]. We begin with a proposition which guarantees that if the faithful, discrete, nonparabolic representations come from the convex real projective structures on a closed hyperbolic 3-manifold, then the limit representation is nonparabolic.

Proposition 6. *Let M be a closed hyperbolic 3-manifold. Suppose ρ_k are discrete faithful holonomy representations corresponding to strictly convex real projective structures on M and $[\rho_k]\to[\tau]$. Then there exists a discrete faithful nonparabolic representation ρ such that $\rho_k\to\rho$ and $g_k\rho g_k^{-1}\to\tau$ for some sequence g_k.*

Proof. Since M is a closed hyperbolic 3-manifold, $\Gamma = \pi_1(M)$ has no center and it is not solvable. Indeed if a nonpositively curved manifold has a solvable fundamental group, it is flat [45]. Since ρ_k is a holonomy representations of strictly convex real projective structures, it is Zariski dense either in $\mathrm{PSL}(4,\mathbb{R})$ or in $\mathrm{PSO}(3,1)$ by Theorem A. So all the hypotheses of Theorem 1 are satisfied. It suffices to show that ρ is nonparabolic in Theorem 1. Suppose ρ in Theorem 1 is parabolic with $\widetilde{\rho_k}\to\rho$ as in the proof of Theorem 1 so that $\rho(\Gamma)$ is contained in the set of matrices of the form

$$\begin{bmatrix} M & 0 \\ 0 & \lambda \end{bmatrix}$$

where M is $(4-k)\times(4-k)$ matrix and $\lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_k), k = 1, 2$. Let Ω_k be a strictly convex domain of S^3 so that $\Omega_k/\widetilde{\rho_k}(\Gamma) = M$. Set C_k be a cone over Ω_k in $\mathbb{R}^4$. Since $\widetilde{\rho_k}(\Gamma)$ preserves a strictly convex cone C_k, $\widetilde{\rho_k}(\Gamma)$ is positively proximal by the Proposition 1.1 of [3]. So two eigenvalues of $\widetilde{\rho_k}(\gamma), \gamma\in\Gamma$ with the largest and the smallest norm are positive. Now consider the group $[\Gamma,\Gamma]$. Then $\rho([\Gamma,\Gamma])$ is of the form

$$\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}$$

where $N \in \mathrm{SL}(4-k,\mathbb{R})$.

Case I) $k = 2$.

In this case $\rho([\Gamma,\Gamma])$ is contained in

$$\begin{bmatrix} \mathrm{SL}(2,\mathbb{R}) & 0 \\ 0 & I \end{bmatrix}.$$

Since $\rho_k(\Gamma)$ is Zariski dense and $[\Gamma,\Gamma]$ is normal in Γ, $\rho_k([\Gamma,\Gamma])$ is normal in a simple group $\mathrm{PSL}(4,\mathbb{R})$ or $\mathrm{PSO}(3,1)$. Its Zariski closure is a normal subgroup, which is either $\mathrm{PSL}(4,\mathbb{R})$ or $\mathrm{PSO}(3,1)$. So $\rho_k([\Gamma,\Gamma])$ is also Zariski dense. Then by the result of [3], $\mathrm{PSL}(4,\mathbb{R})$ has a property that any Zariski dense subgroup H of $\mathrm{PSL}(4,\mathbb{R})$ has a Zariski dense subgroup of H whose elements have the eigenvalues with the same sign.

Note in the list for condition b after Theorem 1.6 of [3], it should read $p = 2, 4$ for $\mathrm{PSL}(p, \mathbb{R})$. Take k_0 large enough so that eigenvalues of $\widetilde{\rho_k}(\gamma)$ and the eigenvalues of $\rho(\gamma)$ are near for each $\gamma \in \Gamma$ and $k > k_0$. Take a Zariski dense subgroup $\rho_{k_0}(\Gamma')$ of $\rho_{k_0}([\Gamma, \Gamma])$ whose elements have the eigenvalues with the same sign. So all the eigenvalues of $\widetilde{\rho_{k_0}}(\gamma), \gamma \in \Gamma'$ are positive. By Lemma 2 of [14], for any free group $\langle a, b\rangle$ and it representation into $\mathrm{SL}(2, \mathbb{R})$, at least one of three $a, b, a^{-1}b^{-1}$ has a negative trace. So choose any elements $\gamma_1, \gamma_2 \in \Gamma'$ so that they generate a free group. Then one of the three elements $\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_1^{-1}\gamma_2^{-1})$ has eigenvalues $(-\lambda_1, -\lambda_2, 1, 1)$ where $\lambda_i > 0$. Since $\widetilde{\rho_k}(\gamma_i),\ k > k_0$ have all nearby eigenvalues, eigenvalues of $\widetilde{\rho_k}(\gamma_i)$ converge to all positive eigenvalues of $\rho(\gamma_i)$. This is a contradiction.

Case II) $k = 1$.

In this case $\rho([\Gamma, \Gamma])$ is contained in the set of matrices of the form

$$\begin{bmatrix} \mathrm{SL}(3, \mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $P : \rho(\Gamma) \to \mathrm{Iso}(\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3))$ be a homomorphism defined in the proof of Theorem 1. It is shown there that $P\rho$ is a discrete faithful representation into $\mathrm{SL}(3, \mathbb{R})$ after passing to a finite index subgroup of Γ. If $P\rho([\Gamma, \Gamma])$ is reducible in $\mathbb{R}^3$, $P\rho([\Gamma, \Gamma])$ is conjugate to a subgroup of $\mathrm{SL}(3, \mathbb{R})$ consisting of matrices either one of forms

$$\begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

Then if we take $P\rho([[\Gamma, \Gamma], [\Gamma, \Gamma]])$, it is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ * & a & b \\ * & c & d \end{bmatrix}$$

where the 2 by 2 block matrix is in $\mathrm{SL}(2, \mathbb{R})$. Then we can use the same method as in Case I. So suppose $P\rho([\Gamma, \Gamma])$ is irreducible in $\mathbb{R}^3$. Then $P\rho([\Gamma, \Gamma])$ is a nonparabolic subgroup of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ by Corollary 1. Then its Zariski closure in $\mathrm{SL}(3, \mathbb{R})$ is reductive [8]. By Lemma 2.6 (a) of [3], $P\rho([\Gamma, \Gamma])$ is proximal. Since $\widetilde{\rho_k}$ is positively proximal, any proximal element in $P\rho([\Gamma, \Gamma])$ is positively proximal. So $P\rho([\Gamma, \Gamma])$ is positively proximal. Then by Proposition 1.1 of [3], $P\rho([\Gamma, \Gamma])$ leaves invariant a strictly convex cone Ω in $\mathbb{R}^3$. Since $[\Gamma,\Gamma]$ is normal in Γ, for any $\gamma \in \Gamma, \alpha \in [\Gamma,\Gamma]$, there exists $\beta \in [\Gamma,\Gamma]$ such that $\gamma\alpha = \beta\gamma$. So $\beta\gamma(\Omega) = \gamma(\Omega)$. This shows that

$$\triangle = \cap_{\gamma\in\Gamma}\gamma(\Omega)$$

is a Γ invariant strictly convex cone in $\mathbb{R}^3$. If $\triangle$ has empty interior, $P\rho([\Gamma,\Gamma])$ would leave invariant a proper subspace of $\mathbb{R}^3$, so cannot be irreducible. So $\triangle$ has nonempty interior in $\mathbb{R}^3$. By Proposition 1 of [44], the action of $P\rho(\Gamma)$ is proper in $\triangle$. Since $P\rho$ is a faithful discrete representation in $\mathrm{SL}(3,\mathbb{R})$ after passing to a finite index subgroup, we can assume Γ is equal to its finite index subgroup and $\triangle/P\rho(\Gamma)$ is a manifold whose fundamental group is Γ. If it is a compact manifold, there should be a nontrivial dilation in $P\rho(\Gamma)$, see Lemma 3.7 (b) of [3]. Since $\mathrm{SL}(3,\mathbb{R})$ cannot have a nontrivial dilation, this does not happen. Since two 3-manifolds M and $\triangle/P\rho(\Gamma)$ have the universal cover $\mathbb{R}^3$ and have the same fundamental group, they should be homotopy equivalent. If $\triangle/P\rho(\Gamma)$ is noncompact, it is not homotopy equivalent to M, again a contradiction. This shows that ρ cannot be parabolic. q.e.d.

We give a lemma before we prove the theorem. This lemma seems known to the experts but not available in references. So we give a sketch of a proof here. When Γ is a finite group, it is proved in [18, Corollary 30.14].

Lemma 6. *Let $\rho, \phi : \Gamma \to \mathrm{SL}(n,\mathbb{R})$ be two representations with the same character. If ρ is irreducible, then they are $\mathrm{GL}(n,\mathbb{R})$-conjugate.*

Proof. Let A be a real group algebra of Γ. Let ρ and ϕ be extensions to A into $M(n,\mathbb{R})$ with the same notations. Let $\chi_\rho = \mathrm{trace} \circ \rho$ be a character of ρ. Then one can show (private communication with Hyman Bass) that the set of Jordan-Hölder factors of the corresponding A-modules, V_ρ and V_ϕ, are isomorphic, counted with multiplicity iff $\chi_\rho = \chi_\phi$.

A proof of the above claim goes as follows:

Step I) Replace ρ and ϕ by the direct sum of their Jordan-Hölder factors. So assume both of them are semisimple.

Step II) Replace A by its quotient by the intersection of $\mathrm{Ker}\rho$ and $\mathrm{Ker}\phi$. This reduces the problem to the case when A is a finite dimensional semi-simple algebra.

Step III) One could even make a base change to the algebraic closure of $\mathbb{R}$ (where it suffices to prove the result) and so assume that A is a finite product of full matrix algebra over $\mathbb{R}$.

Step IV) Finally one can check the result by evaluating the two characters on the minimal central idempotents of A using Burnside lemma, which gives the dimensions of the isotypic components of the two representation modules.

In particular, they have the same number of composition factors. So ϕ is irreducible also. Once we know that both representations are irreducible with the same character, it is known that they are conjugate in $\mathrm{GL}(n, \mathbb{R})$. q.e.d.

The following theorem is a higher-dimensional analogue of the main result of [14]. After this paper was written, a general case was announced in [5].

Theorem 4. *The holonomy map* $h : \mathfrak{B}^0(M) \to \mathfrak{R}$ *is a homeomorphism onto the Teichmüller component if* M *is a closed hyperbolic* 3*-manifold. The same thing is true even for a hyperbolic closed* 3*-orbifold.*

Proof. Denote R the set of representations from $\pi_1(M)$ to $\mathrm{PSL}(4, \mathbb{R})$ and $\pi : R \to \mathfrak{R}$ the natural projection. We know $h(\mathfrak{B}^0(M))$ is open, so it suffices to show that it is closed. Suppose $[\phi_k] \to [\tau]$ in $\mathfrak{R}$ where $[\phi_k] \in h(\mathfrak{B}^0(M))$. Since ϕ_i is Zariski dense either in $\mathrm{PSL}(4, \mathbb{R})$ or in $\mathrm{PSO}(3, 1)$, it is nonparabolic.

Arrange by conjugation that $\phi_i \to \phi$ and $g_k \phi g_k^{-1} \to \tau$ in R with $\pi(\phi_i) = [\phi_i] \in h(\mathfrak{B}^0(M))$ and ϕ is nonparabolic, discrete and faithful by Proposition 6. Then by Corollary 1, ϕ is irreducible in $\mathrm{PSL}(4, \mathbb{R})$. Let $\Omega_i / \phi_i(\pi_1(M))$ be the corresponding projective structures. We may assume that Ω_i is situated in S^3. Then since S^3 is compact, $\Omega_i \to \Omega$ in the Hausdorff topology and Ω is convex. Clearly Ω is left invariant by ϕ. If Ω is a hemisphere or has an empty interior in S^3, ϕ would leave invariant a proper projective subspace, which contradicts to the irreducibility of ϕ. So $\Omega / \phi(\pi_1(M))$ is a manifold homotopy equivalent to M. Actually by [23], it is homeomorphic to M. By Theorem 3 in [5], it is strictly convex. But since $\phi_i \to \phi$, $[\phi] \in h(\mathfrak{B}^0(M))$.

Consider lifts $\widetilde{\phi}_i$ to $\mathrm{SL}(4, \mathbb{R})$ so that $\widetilde{\phi}_i \to \widetilde{\phi}$. Abusing notations, we will denote them by ϕ_i, ϕ again. Then $g_k \phi g_k^{-1} \to \tau$ implies that τ and ϕ have the same character. By the above lemma, τ and ϕ are conjugate by an element in $\mathrm{SL}(4, \mathbb{R}) \cup \mathrm{SL}_-(4, \mathbb{R})$. So ϕ and τ represent the same strictly convex real projective structures on M.

So $h(\mathfrak{B}^0(M))$ is open and closed in $\mathfrak{R}$. This finishes the proof. For the statement of an orbifold, just note that Theorem 1 and Proposition 6 remain true for a group with torsion and the holonomy theorem is also true for an orbifold [40]. q.e.d.

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